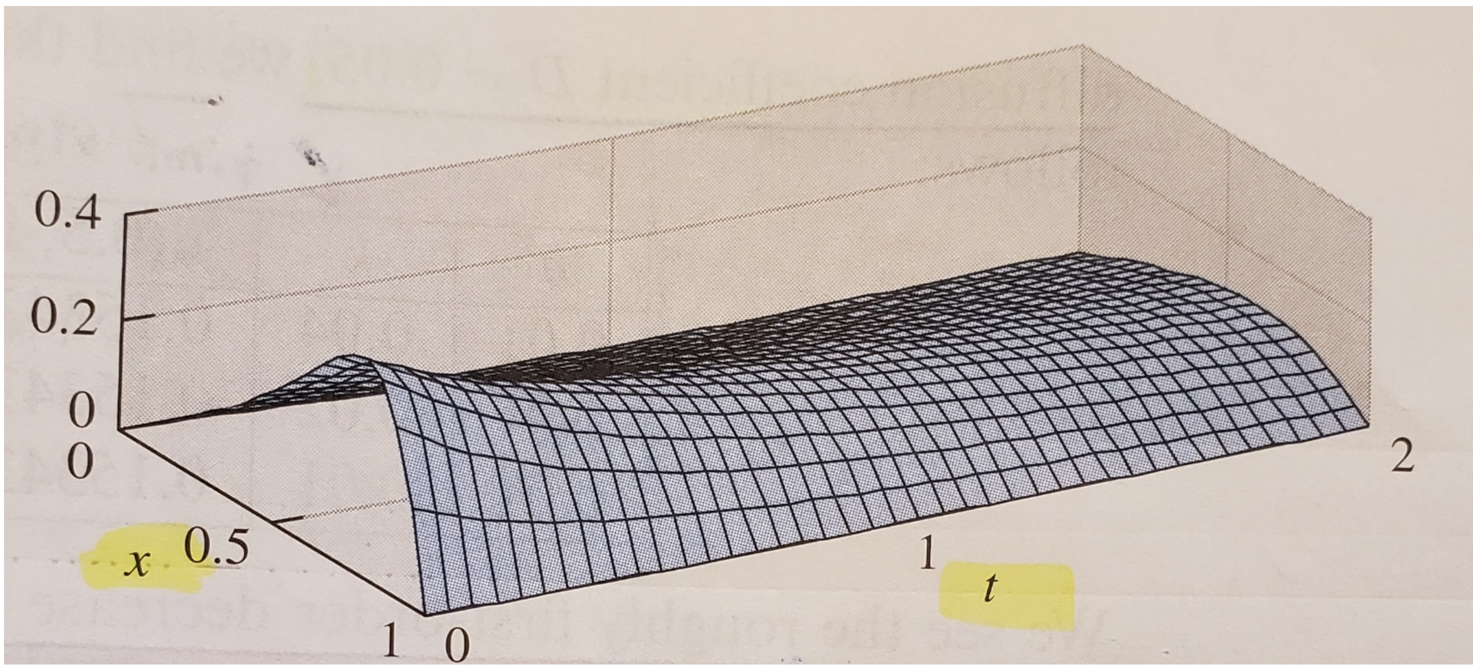
Burgers' Equation

$$u_t + \overbrace{uu_x}^{\text{Non-Linear}} = Du_{xx}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = D\frac{\partial^2 u}{\partial x^2}$$

Example 8.12: Boundary Conditions (Dirichlet)

$$\begin{cases} u_t + uu_x = Du_{xx} \\ u(x,0) = \dfrac{2\pi\beta D \sin \pi x}{\alpha + \beta \cos \pi x} \quad \text{for } 0 \le x \le 1 \\ \left.\begin{array}{l} u(0,t) = 0 \ \ \text{for all } t \ge 0 \\ u(1,t) = 0 \ \ \text{for all } t \ge 0 \end{array}\right\} \text{L, R Boundary} \end{cases}$$



Exact Solution:

$$u(x,t)=\frac{2D\beta\pi e^{-D\pi^2 t}\sin\pi x}{\alpha+\beta e^{-D\pi^2 t}\cos\pi x}$$

$$\therefore$$

$$\frac{\partial u}{\partial t}=\frac{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)\left(2D\beta\pi\sin\pi x\right)\left(e^{-D\pi^2 t}\right)\left(-D\pi^2\right)-\left(2DB\pi e^{-D\pi^2 t}\sin\pi x\right)\left(\beta\cos\pi x e^{-D\pi^2 t}\right)\left(-D\pi^2\right)}{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)^2}$$

$$=\frac{\left(-2\pi^3\alpha\beta D^2 e^{-D\pi^2 t}\sin\pi x-2\pi^3\beta^2 D^2 e^{-2D\pi^2 t}\cos\pi x\sin\pi x\right)+\left(2\pi^3\beta^2 D^2 e^{-2D\pi^2 t}\cos\pi x\sin\pi x\right)}{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)^2}$$

$$=\frac{2\pi^3\beta D^2 e^{-D\pi^2 t}\sin\pi x\left[-\alpha\cancel{-\beta e^{-D\pi^2 t}\cos\pi x}\cancel{+\beta e^{-D\pi^2 t}\cos\pi x}\right]}{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)^2}$$

$$=\frac{2\pi^3\beta D^2 e^{-D\pi^2 t}\sin\pi x\left(-\alpha\right)}{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)^2}$$

$$=\frac{-2\pi^3\alpha\beta D^2 e^{-D\pi^2 t}\sin\pi x}{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)^2}$$

$$\frac{\partial u}{\partial x}=\frac{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)\left(2D\beta\pi e^{-D\pi^2 t}\right)\left(\pi\cos\pi x\right)-\left(2D\beta\pi e^{-D\pi^2 t}\sin\pi x\right)\left(\beta e^{-D\pi^2 t}\right)\left(-\pi\sin\pi x\right)}{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)^2}$$

$$=\frac{\left(2\pi^2\alpha\beta De^{-D\pi^2 t}\cos\pi x+2\pi^2\beta^2De^{-2D\pi^2 t}\cos^2\pi x\right)+\left(2\pi^2\beta^2De^{-2D\pi^2 t}\sin^2\pi x\right)}{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)^2}$$

$$=\frac{2\pi^2\beta De^{-D\pi^2 t}\left[\alpha\cos\pi x+\beta e^{-D\pi^2 t}\cos^2\pi x+\beta e^{-D\pi^2 t}\sin^2\pi x\right]}{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)^2}$$

$$=\frac{2\pi^2\beta De^{-D\pi^2 t}\left[\alpha\cos\pi x+\beta e^{-D\pi^2 t}\left(\cos^2\pi x+\sin^2\pi x\right)\right]}{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)^2}$$

$$=\frac{2\pi^2\alpha\beta De^{-D\pi^2 t}\cos\pi x+2\pi^2\beta^2De^{-2D\pi^2 t}}{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)^2}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)^2\left(2\pi^2\alpha\beta De^{-D\pi^2 t}\right)\left(-\pi\sin\pi x\right)-\left(2\pi^2\alpha\beta De^{-D\pi^2 t}\cos\pi x+2\pi^2\beta^2De^{-2D\pi^2 t}\right)(2)\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)\left(\beta e^{-D\pi^2 t}\right)\left(-\pi\sin\pi x\right)}{\left[\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)^2\right]^2}$$

$$=\frac{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)\cancel{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)}\left(-2\pi^3\alpha\beta De^{-D\pi^2 t}\sin\pi x\right)-\left(2\pi^2\alpha\beta De^{-D\pi^2 t}\cos\pi x+2\pi^2\beta^2De^{-2D\pi^2 t}\right)\cancel{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)}\left(-2\pi\beta e^{-D\pi^2 t}\sin\pi x\right)}{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)^{\cancel{4}3}}$$

$$=\frac{-2\pi^3\alpha^2\beta De^{-D\pi^2 t}\sin\pi x-2\pi^3\alpha\beta^2De^{-2D\pi^2 t}\cos\pi x\sin\pi x-\left[-4\pi^3\alpha\beta^2De^{-2D\pi^2 t}\cos\pi x\sin\pi x-4\pi^3\beta^3De^{-3D\pi^2 t}\sin\pi x\right]}{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)^3}$$

$$=\frac{-2\pi^3\alpha^2\beta De^{-D\pi^2 t}\sin\pi x\cancel{-2\pi^3\alpha\beta^2De^{-2D\pi^2 t}\cos\pi x\sin\pi x}+\cancel{4}2\pi^3\alpha\beta^2De^{-2D\pi^2 t}\cos\pi x\sin\pi x+4\pi^3\beta^3De^{-3D\pi^2 t}\sin\pi x}{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)^3}$$

$$=\frac{-2\pi^3\alpha^2\beta De^{-D\pi^2 t}\sin\pi x+2\pi^3\alpha\beta^2De^{-2D\pi^2 t}\cos\pi x\sin\pi x+4\pi^3\beta^3De^{-3D\pi^2 t}\sin\pi x}{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)^3}$$

$$\therefore u_t + uu_x = Du_{xx}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = D\frac{\partial^2 u}{\partial x^2}$$

LHS:

$$\frac{-2\pi^3\alpha\beta D^2 e^{-D\pi^2 t}\sin\pi x}{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)^2} + \left[\frac{2D\beta\pi e^{-D\pi^2 t}\sin\pi x}{\alpha+\beta e^{-D\pi^2 t}\cos\pi x}\right]\left[\frac{2\pi^2\alpha\beta De^{-D\pi^2 t}\cos\pi x + 2\pi^2\beta^2 De^{-2D\pi^2 t}}{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)^2}\right] =$$

$$\frac{-2\pi^3\alpha\beta D^2 e^{-D\pi^2 t}\sin\pi x}{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)^2} + \frac{4\pi^3\alpha\beta^2 D^2 e^{-2D\pi^2 t}\cos\pi x\sin\pi x + 4\pi^3\beta^3 D^2 e^{-3D\pi^2 t}\sin\pi x}{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)^3} =$$

$$\frac{-2\pi^3\alpha\beta D^2 e^{-D\pi^2 t}\sin\pi x}{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)^2} * \frac{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)}{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)} + \frac{4\pi^3\alpha\beta^2 D^2 e^{-2D\pi^2 t}\cos\pi x\sin\pi x + 4\pi^3\beta^3 D^2 e^{-3D\pi^2 t}\sin\pi x}{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)^3} =$$

$$\frac{-2\pi^3\alpha^2\beta D^2 e^{-D\pi^2 t}\sin\pi x \cancel{- 2\pi^3\alpha\beta^2 D^2 e^{-2D\pi^2 t}\cos\pi x\sin\pi x}}{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)^3} + \frac{\cancel{4}2\pi^3\alpha\beta^2 D^2 e^{-2D\pi^2 t}\cos\pi x\sin\pi x + 4\pi^3\beta^3 D^2 e^{-3D\pi^2 t}\sin\pi x}{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)^3} =$$

$$\frac{-2\pi^3\alpha^2\beta D^2 e^{-D\pi^2 t}\sin\pi x + 2\pi^3\alpha\beta^2 D^2 e^{-2D\pi^2 t}\cos\pi x\sin\pi x + 4\pi^3\beta^3 D^2 e^{-3D\pi^2 t}\sin\pi x}{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)^3} =$$

RHS:

$$= Du_{xx}$$

$$= D\frac{\partial^2 u}{\partial x^2}$$

$$= D\left[\frac{-2\pi^3\alpha^2\beta De^{-D\pi^2 t}\sin\pi x + 2\pi^3\alpha\beta^2 De^{-2D\pi^2 t}\cos\pi x\sin\pi x + 4\pi^3\beta^3 De^{-3D\pi^2 t}\sin\pi x}{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)^3}\right]$$

$$= \frac{-2\pi^3\alpha^2\beta D^2 e^{-D\pi^2 t}\sin\pi x + 2\pi^3\alpha\beta^2 D^2 e^{-2D\pi^2 t}\cos\pi x\sin\pi x + 4\pi^3\beta^3 D^2 e^{-3D\pi^2 t}\sin\pi x}{\left(\alpha+\beta e^{-D\pi^2 t}\cos\pi x\right)^3}$$

$$\therefore 0 = 0$$